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BUCKLING OF A DUCTILE COLUMN UNDER RIGID LOADING

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The problem of buckling of rheological structures has been treated in numerous studies, a review of which can be found in [1, 2]. In the majority of cases, the effect is discussed at constant loads. The present paper studies the possibility of buckling of a column during an increase in compressive force at a specified constant rate of deformation. Use is made of the singular point theory [3], which refines the pseudo-bifurcation approach [4] in problems of buckling of ideal systems during creep.

To defining relationship is taken in the form [5]

$$\dot{p}p^{\alpha} = A\sigma^{n}, \tag{1}$$

where $p = \varepsilon - \sigma/E$ is the creep strain; A, n, and α are the parameters of the medium. We shall consider a material in which the elastic strain σ/E may be neglected, i.e., $E \to \infty$, $p = \varepsilon$.

The equilibrium of a column of length l acted upon by a load T near the rectilinear position obeys equations that follow from the hypothesis of plane sections and the equation of moments about a neutral axis:

$$\int_{\Omega}\Delta pzd\Omega = Jv'', \int\Delta\sigma zd\Omega = -Tv. \tag{2}$$

Here Ω, J are the area and moment of inertia of the cross section; the symbol Δ denotes the increment of the corresponding quantity; v is the deflection of the column; v'' is the second derivative of the deflection with respect to the longitudinal coordinate y.

Small increments of stresses and strains satisfy the linearized equation

$$p^{\alpha}\Delta\dot{p} + \alpha p^{\alpha-1}\dot{p}\Delta p = An\sigma^{n-1}\Delta\sigma. \tag{3}$$

We multiply Eq. (3) by z, integrate over the cross-sectional area Ω, and allowing for (1), (2) and making the substitution $T = \sigma\Omega$, obtain

$$pJ\dot{v}'' + \alpha\dot{p}Jv'' = -np\dot{p}\Omega v. \tag{4}$$

Equation (4) is satisfied by the functions

$$v = U_0\sin\mu y,\ \dot{v} = U_1\sin\mu y,\ \mu = j\pi/l. \tag{5}$$

Substitution of (5) into (4) yields

$$\dot{p}(np/D - \alpha)U_0 - pU_1 = 0, \tag{6}$$

where $D = J\mu^2/\Omega$. The deformation program is specified as linear:

$$p = kt \tag{7}$$

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(k = const being the rate of deformation). Equation (6) is rewritten in the form

$$(\tau - \alpha)U_0 - tU_1 = 0. \tag{8}$$

We introduce the dimensionless time

$$\tau = ktn/D. \tag{9}$$

Equation (8) isolates the value $\tau = \tau_1 = \alpha$ (the Rabotonov–Shesterikov criterion [6]) as a singular point of the process. Actually, a disturbance specifying the value of the initial deflection U_0 up to the instant τ_1 gives a deflection that decreases immediately after the instant of disturbance and increases if $\tau > \tau_1 = \alpha$, as can be judged from the sign of the deflection rate U_1.

To analyze the acceleration of the column's deflection and its higher time derivatives at the instant of disturbance we shall raise the order of the defining relation. For brevity, we shall continue ourselves to the third-order derivative. This is sufficient to explain the relationships which thus arise. We differentiate Eq. (1) twice with respect to time:

$$\ddot{p}p^\alpha + \alpha p^{\alpha-1}\dot{p}^2 = An\sigma^{n-1}\dot{\sigma}; \tag{10}$$

$$p^{(3)}p^\alpha + 3\alpha\dot{p}\ddot{p}p^{\alpha-1} + \alpha(\alpha - 1)\dot{p}^3p^{\alpha-2} = An(\sigma^{n-1}\ddot{\sigma} + (n-1)\sigma^{n-2}\dot{\sigma}^2). \tag{11}$$

By analogy with the way in which the equation for amplitudes (8) was obtained from Eq. (1), we derive from (10) and (11) the equations for the quantities U_i introduced by means of the formula $v^{(i)} = U_i\sin\mu y$ ($i = 0, 1, 2,\ldots$), where $v^{(i)}$ is a time derivative of order i.

Variation of (1) and (11) yields

$$t^2\Delta\ddot{p} + 2\alpha t\Delta\dot{p} + \alpha(\alpha - 1)\Delta p = nkt^2[(n-1)\dot{\sigma}\Delta\sigma/\sigma + \Delta\dot{\sigma}]/\sigma; \tag{12}$$

$$\begin{aligned} &t^3\Delta p^{(3)} + 3\alpha t^2\Delta\ddot{p} + 3\alpha(\alpha - 1)t\Delta\dot{p} + \alpha(\alpha - 1)(\alpha - 2)\Delta p \\ &= nkt^3[(n-1)(n-2)\dot{\sigma}^2\Delta\sigma/\sigma^2 + \ddot{\sigma}\Delta\sigma/\sigma + 2(n-1)\dot{\sigma}\Delta\dot{\sigma}/\sigma + \Delta\ddot{\sigma}]. \end{aligned} \tag{13}$$

When p = kt, it follows from (1) that

$$k^{\alpha+1}t^\alpha = A\sigma^n. \tag{14}$$

Differentiating this equation twice, we obtain

$$\dot{\sigma} = \frac{\alpha}{n}\frac{\sigma}{t}, \ddot{\sigma} = \frac{\alpha}{n}\left(\frac{\alpha}{n} - 1\right)\frac{\sigma}{t^2}.$$

Allowing for the latter expressions, we derive from (12) and (13)

$$\alpha(\tau - \alpha + 1)U_0 + (\tau - 2\alpha)U_1t - U_2t^2 = 0; \tag{15}$$

$$\begin{aligned} &\alpha(\alpha - 1)(\tau - \alpha + 2)U_0 + \alpha(2\tau - 3\alpha + 3)U_1t \\ &+ (\tau - 3\alpha)U_2t^2 - U_3t^3 = 0. \end{aligned} \tag{16}$$

Equations (8), (15) form a system for variables U_0, U_1, U_2. Let the column, as a result of some disturbance receive a given initial magnitude of acceleration of deflection amplitude U_2. The deflection U_0 and its rate U_1 are expressed in terms of U_2 by means of the solution of the system of equations (8), (15):

$$U_0 = t^2U_2B_0/B_2,\ U_1 = tU_2B_1/B_2$$
$$(B_0 = 1,\ B_1 = \tau - \alpha,\ B_2 = \tau^2 - 2\alpha\tau + \alpha(\alpha + 1)).$$

Similarly, if U_3 is given, the initial values of U_0, U_1, U_2 are expressed in terms of U_3 from the solution of the system (8), (15), (16). We write it in the matrix form

$$\begin{vmatrix} \tau - \alpha & -t & 0 \\ \alpha(\tau - \alpha + 1) & (\tau - 2\alpha)t & -t^2 \\ \alpha(\alpha - 1)(\tau - \alpha + 2) & \alpha(2\tau - 3\alpha + 3)t & t^2(\tau - 3\alpha) \end{vmatrix} \begin{vmatrix} U_0 \\ U_1 \\ U_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ t^3 U_3 \end{vmatrix}. \tag{17}$$

From the solution of (17) we have

$$U_0 = t^3 U_3 B_0 / B_3,\ U_1 = t^2 U_3 B_1 / B_3,\ U_2 = t U_3 B_2 / B_3,$$

where $B_3 = \tau^3 - 3\alpha\tau^2 + 3\alpha(\alpha + 1)\tau - \alpha(\alpha + 1)(\alpha + 2)$ is the determinant of the matrix of the system.

A singular point of order N τ_N corresponds to the equality to zero of the determinant of the system of order N. As is evident from the solutions for N = 2 and N = 3, the initial values of the amplitudes U_0, U_1,..., U_{N-1} increase indefinitely. The polynomial B_2 has no roots, and hence, no second-order singular point exists in this problem. A third-order singular point can be found numerically from the solution of the equation $B_3 = 0$. Continuing the process of setting up the polynomials according to the scheme discussed, we derive the following:

$$B_4 = \tau^4 - 4\alpha\tau^3 + 6\alpha(\alpha + 1)\tau^2 - 4\alpha(\alpha + 1)(\alpha + 2)\tau + \alpha(\alpha + 1)(\alpha + 2)(\alpha + 3).$$

It is easy to check the general formula

$$B_N = \sum_{i=0}^{N} (-1)^{N+i} \tau^i C_i^N (\alpha)_{N-i},$$

where C_i^N are binomial coefficients; $(\alpha)_j = \alpha(\alpha + 1) \ldots (\alpha + j - 1)$ is Pochgammer's symbol; $(\alpha)_0 = 1$. In addition, the following differential relation may be used in obtaining the roots:

$$B'_N = NB_{N-1}.$$

Numerical calculation shows that the even polynomials have no roots, and the roots of the odd polynomials increase steadily as α increases and are ordered in accordance with the orders of the polynomials (see Fig. 1).

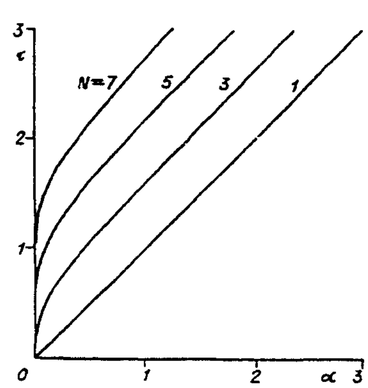

Fig. 1

To a singular point of order N corresponds a critical time t_N given by Eq. (9):

$$t_N = \frac{\tau_N}{kn} D. \tag{18}$$

The stress in the column during that time increases from zero to a value which we find from (14):

$$\sigma = \left(\frac{k}{A}\right)^{1/n} \left(\frac{\tau_N D}{n}\right)^{\alpha/n}.$$

The solution obtained can, for example, be used to estimate the buckling-safe rate of load applications in a buckling experiment at a constant load σ_*. In order for buckling to be avoided at the stage of load increase from zero to the working value σ_*, the time of increase of the effort should be smaller than the time corresponding to the first singular point t_1. Hence, according to (14) and (18), we have

$$k > A\sigma_*^n \left(\frac{\alpha D}{n}\right)^{-\alpha}.$$

We shall compare the proposed approach to the results that follow from certain known conditional criteria of stability in creep. We note at once that these criteria were developed for the case of a constant load and a medium having a certain modulus of elasticity E. The transfer of the conditional criteria to the analysis of the loading $p = kt$ of an instantaneously rigid material is somewhat formal.

Tangent Module Criterion [7]. Eliminating the time t from Eq. (14) and from the loading law (7), we obtain the equation of an isochrone, $kp^\alpha = A\sigma^n$. We shall differentiate it with respect to p. We find the tangent modulus $E_c = \alpha\sigma/(np)$. The critical time will be determined from the Euler formula with modulus E_c instead of E: $\sigma = E_c D$. Hence

$$t_{\mathrm{Shml}} = \frac{\alpha}{nk} D.$$

Critical Strain Criterion [8]. For an elastic column, the strain corresponding to the Euler load σ_e is $\varepsilon = \sigma_e/E = D$. In our case, $p = D$, whence there immediately follows

$$t_{\mathrm{J}} = \frac{D}{k}.$$

Here the critical time is independent of the properties of the medium.

Disturbance Relaxation Criterion [9]. When the column is at equilibrium in a deflected state acted upon by some disturbing load (of moment m), the second equation (2) should be replaced by

$$\int_\Omega \Delta\sigma z d\Omega = -Tv - m. \tag{19}$$

As the critical time we take the time in which at a constant deflection v, the moment m sustaining this state will drop to zero. Considering that $v = \text{const}$, from (2), (3), (19) with $\dot{p} = k$ we obtain the equation for the amplitude M of the disturbing moment ($m = M\sin\mu y$):

$$M = U_0 T(D\alpha/(tnk) - 1).$$

The moment decreases from an infinitely large value at $t = 0$ to zero in a time

$$t_{\mathrm{Iv}} = \frac{\alpha}{nk} D,$$

which coincide in magnitude with t_{Shnl} and with the first-order singular point t_1.

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